

Graphs with no $2\delta + 1$ cycle

Galen E. Turner III

Mathematics and Statistics Program
Louisiana Tech University, Ruston, Louisiana
`gturner@coes.LaTech.edu`

Submitted: 9 July 2003

MR Subject Classifications: 05C35, 05C38, 05C75

Abstract

Dirac proved that any graph with minimum vertex degree δ contains either a cycle of length at least 2δ or a Hamilton cycle. Motivated by this result, we characterize those graphs having no cycle longer than 2δ .

1 Introduction

Dirac [1952] proved that any 2-connected graph with minimum vertex degree δ contains either a cycle of length at least 2δ or a Hamilton cycle. In this paper, we shall characterize those graphs with minimum vertex degree δ which have no cycle of length greater than 2δ . This characterization was also motivated by Ali and Staton [1996] in which similar results were given for graphs with no path exceeding $2\delta + 1$. In particular, as a case in their work, they prove that if the number of vertices of a non-hamiltonian 2-connected graph is exactly $2\delta + 1$, then the graph must be isomorphic to the join of a graph on δ vertices and a totally disconnected graph on $\delta + 1$ vertices. Here, we will show that this same type of structure is present when the number of vertices exceeds $2\delta + 1$.

2 Preliminaries

Unless specified otherwise, the terminology used here will follow Bondy and Murty [1976]. In particular, a graph may have loops and parallel edges. If v is a vertex of a graph G , then the set of neighbors of v will be denoted $N(v)$ or $N_G(v)$, and the minimum vertex degree of G will be denoted δ or $\delta(G)$. If S is a subset of the vertex set of G , then $G[S]$ denotes the induced subgraph of G on S . The totally disconnected graph on m vertices will be denoted N_m , and any subgraph of K_n will be denoted H_n . The *join* of two graphs G and H is the graph $G \vee H$ and is obtained by taking disjoint copies of G and H and adding edges joining every vertex of G to every vertex of H . In particular, if $|V(G)| = n$

and $|V(H)| = m$, then the graph $G \vee H - (E(G) \cup E(H))$ is isomorphic to $K_{n,m}$. Finally, if $x_1x_2 \dots x_n$ is a path P in a graph G , then $P[x_i, x_j]$, $P(x_i, x_j)$, $P(x_i, x_j]$, and $P(x_i, x_j)$, will denote the subpaths $x_ix_{i+1} \dots x_j$, $x_ix_{i+1} \dots x_{j-1}$, $x_{i+1}x_{i+2} \dots x_j$, and $x_{i+1}x_{i+2} \dots x_{j-1}$, respectively.

3 The Main Theorem

To establish the main result of this paper, we shall use the following theorem of Dirac [1952].

Theorem 3.0.1 *Let G be a simple 2-connected graph with minimum vertex degree δ and suppose $|V(G)| \geq 3$. Then G contains either a cycle of length at least 2δ or a Hamilton cycle.*

Motivated by this result, we will now prove the following characterization for the graphs having no cycle longer than 2δ .

Theorem 3.0.2 *Let G be a simple 2-connected graph with minimum vertex degree δ . If G has no cycle of length at least $2\delta + 1$, then G is Hamiltonian or $G \cong H_\delta \vee N_m$ where $m > \delta$.*

Proof. Let G be a graph with no Hamilton cycle and no cycle of length at least $2\delta + 1$. By Theorem 3.0.1, G has a cycle C of length 2δ . Now, since G is not Hamiltonian, there is a vertex z in $V(G) - V(C)$, and by Menger's Theorem [1927], there are 2 paths Q_1 and Q_2 from z to C that have only the vertex z in common and such that each Q_i meets C in exactly one vertex z_i .

Lemma 3.0.3 *Q_1 and Q_2 must not meet consecutive vertices on C .*

Proof. Suppose Q_1 and Q_2 meet C at z_1 and z_2 where z_1z_2 is an edge of C . Then G has a cycle C' formed by the subpath $C - \{z_1z_2\}$ together with the path z_1zz_2 . Therefore, C' contains $2\delta + 1$ vertices of G , namely, all the vertices in $V(C) \cup \{z\}$; a contradiction. \square

Since there are paths from every vertex of $V(G) - V(C)$ to C , choose a longest path P subject to the condition that one of the endvertices of P lies on C and no other vertex of P lies on C . We label the endvertex of P that is not in $V(C)$ as x and the vertex in $V(P) \cap V(C)$ is labelled y . Now, since P is a longest path of this type, it is clear that every neighbor of x lies in $V(P) \cup V(C)$. Moreover, we have the following lemma.

Lemma 3.0.4 *x has a neighbor in $V(C) - y$.*

Proof. Suppose not. Then every neighbor of x lies on P . Since G is 2-connected, it is clear that there is a vertex of P adjacent to some vertex on C . Viewing P as a directed path from x to y , let z be the first vertex on P that is adjacent to a vertex, z' , of C , and

let u_x be the first neighbor of x after z on P . Now, C is partitioned into two paths from z' to y , and we arbitrarily label them C_1 and C_2 . Since C has size exactly 2δ , one of C_1 and C_2 has at least δ edges. Without loss, assume that C_1 has at least δ edges.

Now, let P' be the path containing $z'z$, $P[z, x]$, xu_x , and $P[u_x, y]$. Notice that P' contains all the vertices of P except those on $P(z, u_x)$. Since there are no neighbors of x on $P(z, u_x)$, it is clear that in addition to x itself, P' contains all the neighbors of x ; thus, P' contains at least $\delta + 1$ vertices from P as well as the vertex z' on C . Since C_1 contains $\delta + 1$ vertices, we can combine C_1 and P' to obtain a cycle C' of size $|V(C_1)| + |V(P')| - 2 \geq (\delta + 1) + (\delta + 2) - 2 = 2\delta + 1$; a contradiction. \square

Having proved Lemma 3.0.4, let the vertices of $C - y$ that are adjacent to x be labelled x_1, x_2, \dots, x_k in cyclic order on C , where y lies between x_k and x_1 on C , and relabel y as x_{k+1} . It is clear that the cycle C is partitioned into $k + 1$ internally disjoint paths $C(x_1), C(x_2), \dots, C(x_{k+1})$ where $C(x_i)$ is the subpath of C from x_i to x_{i+1} that contains no other vertex in $\{x_1, x_2, \dots, x_{k+1}\}$; here $C(x_{k+1})$ is the subpath of C from x_{k+1} to x_1 . If $k = 1$, then C can be partitioned into two paths from x_1 to x_2 . We arbitrarily label one of these $C(x_1)$ and the other $C(x_2)$. Since G has no cycle containing $2\delta + 1$ vertices, it is clear by Lemma 3.0.3 that Each $C(x_i)$ contains at least one vertex in its interior, and we let the vertex of $C(x_i)$ that is adjacent to x_i be labelled a_i .

Lemma 3.0.5 *There is no path from a_i to a_j avoiding $x \cup C$.*

Proof. Suppose Q is a path joining a_i to a_j and avoiding $x \cup C$. It is easy to see that G has a cycle C' formed by the edges of $C - \{a_j x_j, x_i a_i\}$ together with the path $x_j x x_i$ and the edges of Q . Thus, C' contains the vertices of C and x ; a contradiction. \square

Now, consider $C(x_k)$ and $C(x_{k+1})$. These paths have the vertex x_{k+1} as an endvertex. If one of these paths has less than $|P|$ internal vertices, then we can delete the path from C and replace it with the path P and the edge xx_k or xx_1 creating a longer cycle. Thus, each of $C(x_k)$ and $C(x_{k+1})$ has length at least $|P| + 1$. So, the length of $C(x_j)$ is greater than or equal to $|P| + 1$ for $j \in \{k, k + 1\}$, and by Lemma 3.0.3, the length of $C(x_i)$ is at least 2 for $i \in \{1, \dots, k - 1\}$. This implies that $|C| \geq 2(|P| + 1) + (k - 1)2$ which means that $2\delta \geq 2|P| + 2k$. Therefore, $\delta \geq |P| + k$.

Now, since $d_G(x) \geq |P| + k$ and x is adjacent to exactly k vertices in $V(C) - x_{k+1}$, the vertex x must be adjacent to every vertex on $P - x$. This implies that $d_G(x) = |P| + k$ which in turn means that $\delta = |P| + k$. Furthermore, this restriction shows that each $C(x_i)$ has length 2 when $i \in \{1, \dots, k - 1\}$ and that both $C(x_k)$ and $C(x_{k+1})$ have length $|P| + 1$.

Lemma 3.0.6 $|P| = 1$. Moreover, $\delta = k + 1$ and $N_G(x) = \{x_1, \dots, x_{k+1}\}$.

Proof. Suppose $|P| > 1$ and let the vertices of P from x to x_{k+1} be labelled $u_1, u_2, \dots, u_{|P|}$ in this order where $u_{|P|} = x_{k+1}$. Since xu_{i+1} is an edge, we have a path $P_i = P - \{u_i u_{i+1}\} \cup xu_{i+1}$ for each u_i , and this path is a longest path having one endvertex on C and the other end being u_i . Since $d_G(u_i) \geq \delta$, the vertex u_i must be adjacent to at least k vertices on

$C - x_{k+1}$, and using Lemmas 3.0.3 and 3.0.5, it is easy to see that the neighbors of u_i on $C - x_{k+1}$ must be exactly the vertices in $\{x_1, x_2, \dots, x_k\}$. So, u_i must be adjacent to every member of P_i . But, $(C - \{a_1\}) \cup \{x_2 x u_1 x_1\}$ is a $2\delta + 1$ cycle, since $u_1 \neq x_{k+1}$. \square

Now, by Lemma 3.0.6, every path from a vertex in $V(G) - V(C)$ to $V(C)$ must have length 1; so, since G is connected, every vertex in $V(G) - V(C)$ has a neighbor in $V(C)$. If x' and x'' are distinct members of $V(G) - V(C)$, then $x'x''$ is not an edge of G ; otherwise if y is a neighbor of x' on C then $yx'x''$ is a path of length 2 having only one common vertex with C . We conclude that $G[V(G) - V(C)]$ is totally disconnected.

Upon combining Lemmas 3.0.3, 3.0.5, and 3.0.6, we see that $N(x) = \{x_1, \dots, x_{k+1}\}$ for every x in $V(G) - V(C)$. Moreover, the set of neighbors for each a_i is $\{x_1, \dots, x_{k+1}\}$ since $a_i a_j$ is not an edge of G and $\delta = k + 1$. Therefore, $G[\{a_1, \dots, a_{k+1}\} \cup (V(G) - V(C))]$ is totally disconnected with $k + 1 + |V(G) - V(C)| = m$ vertices. Since $V(G) - V(C)$ has at least one vertex, m exceeds δ , and the theorem is established. \square

Corollary 3.0.7 *Let G be a 2-connected graph with at least $2\delta + 1$ vertices. Then, G has no cycle of length at least $2\delta + 1$ if and only if $G \cong H_\delta \vee N_m$ where $m > \delta$.*

References

- [1996] Ali, A.A. and W. Staton, On extremal graphs with no long paths *Electron. J. Combin.*, **3** (1996), no. 1, Research Paper 20.
- [1976] Bondy, J.A. and U.S.R. Murty, *Graph Theory with Applications*. North Holland, New York, 1976.
- [1952] Dirac, G.A., Some Theorems on Abstract Graphs. *Proc. Lond. Math. Soc.* **27** (1952), 69–81.
- [1927] Menger, K., Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 96–115.